Lecture Note

Topic Name: series solution

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→ Introduction:-

→ classification of points

consider second order diff. eqn

$$a_2(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_0(x)y=0$$

For $x \in I \in \mathbb{R}$, $a_2(x) \neq 0$

$$\frac{d^2y}{dx^2}+\frac{a_1(x)}{a_2(x)}\frac{dy}{dx}+\frac{a_0(x)}{a_2(x)}y=0 \rightarrow (1)$$

$$\Rightarrow y''+p(x)y'+q(x)y=0 \rightarrow (2)$$

where $p(x)=\dfrac{a_1(x)}{a_2(x)}$, $q(x)=\dfrac{a_0(x)}{a_2(x)}$

From eqn (1) and (2)

1. If $a_2(x) \neq 0$ at $x=x_0$ then x_0 is called an ordinary point.

   i.e. $p(x)$ and $q(x)$ both are an analytic at an ordinary point $x=x_0$

2. If $a_2(x)=0$ at $x=x_0$ then x_0 is called singular point.

   i.e. either $p(x)$ or $q(x)$ or both are not analytic at $x=x_0$

- In case, $x = x_0$ is singuler pt and if $(x-x_0)p(x)$ and $(x-x_0)^2 q(x)$ both are analytic at $x = x_0$ then x_0 is called a Reguler singuler point.

- But $x = x_0$ is singuler pt and if $(x-x_0)p(x)$ and $(x-x_0)^2 q(x)$ not analytic (both or one of them) at $x = x_0$ then x_0 is called Irreguler singuler point.

Examples: 1 to 4.

classify ordinary point, singular pt, regular singular points and Irregular singular pts (if any) of the following diff. eqn

Ex: 1 $\quad y'' + 2x^2 y' - 4xy = 0$

soln Here $y'' + p(x)y' + q(x)y = 0$

i.e. $p(x) = 2x^2$, $q(x) = -4x$

it is clear that $p(x)$ and $q(x)$ are analytic for $\forall x \in \mathbb{R}$ (polynomial)

i.e. For all $x \in \mathbb{R}$ are Ordinary pts. #

Ex:2 $\quad y'' - xy = 0$

Sol^n compare with $y'' + p(x)y' + q(x)y = 0$

i.e. $p(x) = 0$, $q(x) = -x$

Here clear that $p(x)$ and $q(x)$ are analytic for any point of $\mathbb{R}$.

i.e. All $x \in \mathbb{R}$ are ordinary points.

There is no singular pts and so no any pts Regular or Irregular singular pts. #

Ex:3 $\quad x(1-x)y'' + 6x^2 y' - 2y = 0$

Sol^n Given that

$$y'' + \frac{6x^2}{x(1-x)} y' - \frac{2}{x(-x+1)} y = 0$$

$$\Rightarrow y'' + \frac{6x}{1-x} y' - \frac{2}{x(1-x)} y = 0$$

Compare with $y'' + p(x)y' + q(x)y = 0$

i.e. $p(x) = \dfrac{6x}{1-x}$, $q(x) = -\dfrac{2}{x(1-x)}$

Here note that

$p(x) = \dfrac{6x}{1-x}$ is not analytic at $x=1$.

$q(x) = -\dfrac{2}{x(1-x)}$ is not analytic at $x=0,1$

that means

$x = 0, 1$ are singular points

and $x \in \mathbb{R} - \{0, 1\}$ are ordinary points

- To check that $x = 0, 1$ are Regular singular pt(s) and/or Irregular singular points.

→ For singular pt $x_0 = 0$.

$$(x - x_0)\, p(x) = (x - 0)\, \frac{6x}{1-x}$$

$$= \frac{6x^2}{1-x}$$ is analytic at $x = 0$. (because $1 - x \neq 0$ at 0)

and $$(x - x_0)^2 q(x) = -(x-0)^2 \cdot \frac{2}{x(x+1)}$$

$$= -\frac{2x}{1-x}$$ is analytic at $x = 0$ ($\because 1 - x \neq 0$ at 0)

Hence $x = 0$ is Regular singular point.

→ For singular pt $x_0 = 1$.

$$(x - x_0)\, p(x) = (x - 1) \cdot \frac{6x}{1-x} = -6x$$ is an analytic at $x = 1$.

and
$$(x - x_0)^2 q(x) = -\frac{2(1-x)}{x}$$ is analytic at $x_0 = 1$ [$\because x \neq 0$ at $x_0 = 1$]

Hence $x = 1$ is Regular singular point.

i.e. $x = 0, 1$ are singular and Regular singular point and $\forall x \in \mathbb{R} - \{0, 1\}$ are ordinary pts. #

Ex: 4 $\quad 2x^3(x+1)y'' + 4x^3y' + y = 0$

soln Here given that

$$y'' + \frac{4x^3}{2x^3(1+x)}y' + \frac{1}{2x^3(x+1)}y = 0$$

$$\therefore \quad p(x) = \frac{2}{x+1}, \quad q(x) = \frac{1}{2x^3(x+1)}$$

$p(x)$ is not analytic for $x = -1$

$q(x)$ is not analytic for $x = 0, -1$

i.e. $x = 0, -1$ are singular pts and $\forall x \in \mathbb{R} - \{0, -1\}$ are ordinary points.

→ For singular point $x_0 = 0$

$$(x - x_0)p(x) = (x-0)\frac{2}{x+1} = \frac{2x}{x+1}$$

is an analytic at 0. [$x+1 \neq 0$ at 0]

and $(x - x_0)^2 q(x) = (x-0)^2 \cdot \frac{1}{2x^3(x+1)}$

$= \frac{1}{2x(x+1)}$ is not-analytic at $x_0 = 0$

Hence $x_0 = 0$ is Irregular singular pt

→ For singular point $x_0 = -1$

$$(x - x_0)p(x) = (x+1)\frac{2}{x+1} = 2$$

is a constant being analytic at $x = -1$.

$$(x - x_0)^2 q(x) = (x+1)^2 \frac{1}{2x^3(x+1)} = \frac{x+1}{2x^3}$$

is an analytic at $x_0 = -1$

$\therefore$ $x_0 = -1$ is Regular singular pt.

Hence all $x \in \mathbb{R} - \{0, -1\}$ are ordinary points and $x = 0$ is an Irregular pt and $x = -1$ is a Regular singular pt.

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\# Examples for practice

→ classify the pts (ordinary, Regular, Irregular)

1. $x^2 y'' + x y' + x^2 y = 0$
2. $y'' + x^2 y = 0$
3. $(1 - x^2) y'' - 2x y' + 2y = 0$

Note: We will discuss in next lecture, a method to find series solution of ordinary diff. eqn near ordinary point.

!!! All the best !!!