Fourier Integral

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→ $$f(x) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(t) \cos \omega(t-x)\, dt\, d\omega$$

i.e. $$f(x) = \int_0^\infty [A(\omega) \cos \omega x + B(\omega) \sin \omega x]\, d\omega$$

where $$A(\omega) = \frac{1}{\pi} \int_{-\infty}^\infty f(t) \cos \omega t\, dt$$

and $$B(\omega) = \frac{1}{\pi} \int_{-\infty}^\infty f(t) \sin \omega t\, dt$$

known as Fourier integral of $f(x)$.

→ $$f(x) = \int_0^\infty A(\omega) \cos \omega x\, d\omega$$

where $$A(\omega) = \frac{2}{\pi} \int_0^\infty f(t) \cos \omega t\, dt$$

known as Fourier cosine integral of $f(x)$.

→ $$f(x) = \int_0^\infty B(\omega) \sin \omega x\, d\omega$$

where $$B(\omega) = \frac{2}{\pi} \int_0^\infty f(t) \sin \omega t\, dt$$

known as Fourier sine integral of $f(x)$.

Ex: Find the Fourier integral representation of

$$f(x) = \begin{cases} x \; ; & \text{for } 0 \le x \le 2 \\ 0 \; ; & \text{otherwise} \end{cases}$$

Sol^n: The Fourier integral representation of $f(x)$ is given by

$$f(x) = \int_0^\infty [A(\omega) \cos \omega x + B(\omega) \sin \omega x] \, d\omega \quad \text{— (1)}$$

where $A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos \omega t \, dt$

$$= \frac{1}{\pi} \left[\int_{-\infty}^{0} 0 \cos \omega t \, dt + \int_0^2 x \cos \omega t \, dt + \int_2^\infty 0 \cos \omega t \, dt \right]$$

$$= \frac{1}{\pi} \int_0^2 t \cos \omega t \, dt$$

$$= \frac{1}{\pi} \left[(t) \left(\frac{\sin \omega t}{\omega} \right) - (1) \left(- \frac{\cos \omega t}{\omega^2} \right) \right]_0^2$$

$$= \frac{1}{\pi} \left[\left\{ (2) \left(\frac{\sin 2\omega}{\omega} \right) + \left(\frac{\cos 2\omega}{\omega^2} \right) \right\} - \left\{ 0 + \left(\frac{-1}{\omega^2} \right) \right\} \right]$$

$$= \frac{1}{\pi} \left[\frac{2 \sin 2\omega}{\omega} + \frac{\cos 2\omega}{\omega^2} + \frac{1}{\omega^2} \right]$$

$$B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin \omega t \, dt$$

$$= \frac{1}{\pi} \left[\int_{-\infty}^{0} 0 \sin \omega t \, dt + \int_{0}^{2} t \sin \omega t \, dt + \int_{2}^{\infty} 0 \sin \omega t \, dt \right]$$

$$= \frac{1}{\pi} \int_{0}^{2} t \sin \omega t \, dt$$

$$= \frac{1}{\pi} \left[(t)\left(-\frac{\cos \omega t}{\omega}\right) - (1)\left(-\frac{\sin \omega t}{\omega^2}\right) \right]_0^2$$

$$= \frac{1}{\pi} \left[\left\{ (2)\left(-\frac{\cos 2\omega}{\omega}\right) + \frac{\sin 2\omega}{\omega^2} \right\} - \{0 - 0\} \right]$$

$$= \frac{1}{\pi} \left[-\frac{2 \cos 2\omega}{\omega} + \frac{\sin 2\omega}{\omega^2} \right]$$

substituting these values in (1), we have

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \left[\frac{2\omega \sin 2\omega + \cos 2\omega - 1}{\omega^2} \right] \cos \omega x \, d\omega$$

$$+ \frac{1}{\pi} \int_0^{\infty} \left[\frac{\sin 2\omega - 2\omega \cos 2\omega}{\omega^2} \right] \sin \omega x \, d\omega$$

which is required Fourier integral representation of $f(x)$.

Ex: Show that $\int_0^\infty \frac{\cos\omega x + \omega \sin\omega x}{1+\omega^2}\, d\omega = \begin{cases} 0 \; ; & \text{if } x<0 \\ \pi/2 \; ; & \text{if } x=0 \\ \pi e^{-x} \; ; & \text{if } x>0 \end{cases}$

Solⁿ: Consider $f(x) = \begin{cases} 0 \; ; & \text{if } x<0 \\ \pi/2 \; ; & \text{if } x=0 \\ \pi e^{-x} \; ; & \text{if } x>0 \end{cases}$

The Fourier integral of $f(x)$ is

$$f(x) = \frac{1}{\pi}\int_0^\infty [A(\omega)\cos\omega x + B(\omega)\sin\omega x]\, d\omega \quad \text{---} (1)$$

where $A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\omega t\, dt$

$$= \frac{1}{\pi}\left[\int_{-\infty}^{0} 0 \cos\omega t\, dt + \int_0^\infty \pi e^{-t}\cos\omega t\, dt\right]$$

$$= \frac{1}{\cancel{\pi}}\int_0^\infty \cancel{\pi} e^{-t}\cos\omega t\, dt$$

$$= \left[\frac{e^{-t}}{(-1)^2+\omega^2}(-\cos\omega t + \omega\sin\omega t)\right]_0^\infty$$

$$= \left[\{0\} - \left\{\frac{1}{1+\omega^2}(-1+0)\right\}\right]$$

$$= \frac{1}{1+\omega^2}$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\omega t\, dt$$

$$= \frac{1}{\pi}\left[\int_{-\infty}^{0} 0 \sin\omega t\, dt + \int_0^\infty \pi e^{-t}\sin\omega t\, dt\right]$$

$$= \frac{1}{\pi} \int_0^\infty \pi \, e^{-t} \sin \omega t \, dt$$

$$= \left[\frac{e^{-t}}{(-1)^2 + \omega^2} (-\sin \omega t - \omega \cos \omega t) \right]_0^\infty$$

$$= \left[\{0\} - \left\{ \frac{1}{1+\omega^2} (0 - \omega) \right\} \right]$$

$$= \frac{\omega}{1+\omega^2}$$

From ①

$$f(x) = \int_0^\infty \left(\frac{1}{1+\omega^2} \cos \omega x + \frac{\omega}{1+\omega^2} \sin \omega x \right) d\omega$$

$$= \int_0^\infty \left(\frac{\cos \omega x + \omega \sin \omega x}{1+\omega^2} \right) d\omega \quad \text{---} ②$$

At $x = 0$, which is a point of discontinuity of $f(x)$, the value of f^n is

$$f(0) = \frac{1}{2} \left[f(0-0) + f(0+0) \right]$$

where $f(0-0) = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 0 = 0$

$$f(0+0) = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \pi e^{-x} = \pi$$

$$\therefore f(0) = \frac{1}{2} \left[0 + \pi \right]$$

$$= \frac{\pi}{2}$$

From (2)

$$\int_0^\infty \left(\frac{\cos \omega x + \omega \sin \omega x}{1+\omega^2} \right) d\omega = \begin{cases} 0 ; & \text{if } x<0 \\ \pi/2 ; & \text{if } x=\pi \\ \pi e^{-x} ; & \text{if } x>0 \end{cases}$$

Ex: Find the Fourier integral representation of the function $f(x) = \begin{cases} 1 ; & \text{if } |x| \leq 1 \\ 0 ; & \text{if } |x| > 1. \end{cases}$

Hence evaluate $\int_0^\infty \frac{\sin \omega \cos \omega x}{\omega} d\omega$.

Soln: The Fourier integral of $f(x)$ is

$$f(x) = \int_0^\infty [A(\omega) \cos \omega x + B(\omega) \sin \omega x] \, d\omega \quad \text{—(1)}$$

where $A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos \omega t \, dt$

$$= \frac{1}{\pi} \left[\int_{-\infty}^{-1} 0 \cos \omega t \, dt + \int_{-1}^{1} 1 \cos \omega t \, dt + \int_{1}^{\infty} 0 \cos \omega t \, dt \right]$$

$$= \frac{1}{\pi} \int_{-1}^{1} \cos \omega t \, dt$$

$$= \frac{2}{\pi} \int_0^1 \cos \omega t \, dt \quad (\because \cos \omega t \text{ is an even fn})$$

$$= \frac{2}{\pi} \left[\frac{\sin \omega t}{\omega} \right]_0^1$$

$$= \frac{2}{\pi} \left[\frac{\sin \omega}{\omega} \right]$$

$$= \frac{2\sin\omega}{\pi\omega}$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\omega t\, dt$$

$$= \frac{1}{\pi}\left[\int_{-\infty}^{-1} 0 \sin\omega t\, dt + \int_{-1}^{1} 1\cdot\sin\omega t\, dt + \int_{1}^{\infty} 0 \sin\omega t\, dt\right]$$

$$= \frac{1}{\pi}\int_{-1}^{1} \sin\omega t\, dt$$

$$= 0 \qquad (\because \sin\omega t \text{ is an odd } f^n.)$$

From (1)

$$f(x) = \int_0^{\infty}\left[\left(\frac{2\sin\omega}{\pi\omega}\right)\cos\omega x + 0\right] d\omega$$

$$= \frac{2}{\pi}\int_0^{\infty} \frac{\sin\omega\cos\omega x}{\omega}\, d\omega \quad \text{--- (2)}$$

which is Fourier integral of $f(x)$.

From (2)

$$\int_0^{\infty} \frac{\sin\omega\cos\omega x}{\omega}\, d\omega = f(x)$$

$$= \begin{cases} \pi/2 ; & \text{if } |x| \leq 1 \\ 0 ; & \text{if } |x| > 1 \end{cases}$$

Here $x = 1$ is a point of discontinuity of $f(x)$

$$\therefore f(0) = \frac{1}{2}\left[f(1-0) + f(1+0)\right]$$

$$f(1-0) = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 1 = 1$$

$$f(1+0) = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 0 = 0$$

$$\therefore \quad f(0) = \frac{1}{2}\left[1 + 0\right] = \frac{\pi}{2}$$

So, re-write ② as

$$\int_0^\infty \frac{\sin\omega \cos\omega x}{\omega}\, d\omega = \begin{cases} \pi/2 \ ; & \text{if } |x| < 1 \\ \pi/4 \ ; & \text{if } x = 1 \\ 0 \ ; & \text{if } |x| > 1 \end{cases}$$

Taking $x = 0$, we get

$$\int_0^\infty \frac{\sin\omega \cos 0}{\omega}\, d\omega = \pi/2 \qquad (\because 0 \in (-1, 1))$$

$$\therefore \int_0^\infty \frac{\sin\omega}{\omega}\, d\omega = \pi/2$$